Physics 3A: Physics for the Life Sciences

Chapter 2: Motion Along a Line — Differentials, Integrals, Free Fall

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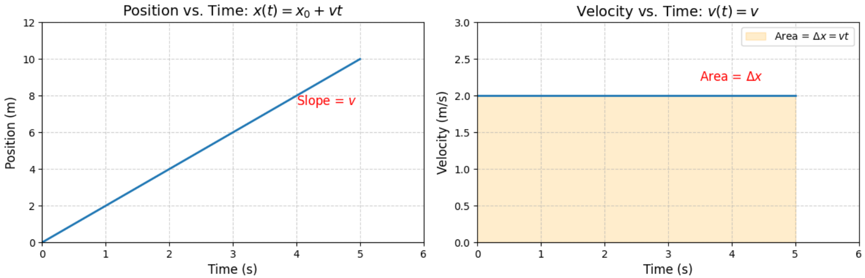
When acceleration is zero, velocity is constant and position changes at a constant rate:

$$v = \frac{\Delta x}{\Delta t} = \frac{x(t) - x(t_\circ)}{t - t_\circ} \longrightarrow x(t) = x(t_\circ) + v(t - t_\circ) \quad \Rightarrow \quad \boxed{x(t) = x_0 + vt}$$



The **slope** of the $x(t)$ vs. t graph gives v_0: A steeper slope means faster motion.

The **area** under the $v(t)$ vs. t graph gives the displacement: here the area is a **rectangle** with height v and width t: $\Delta x = vt$

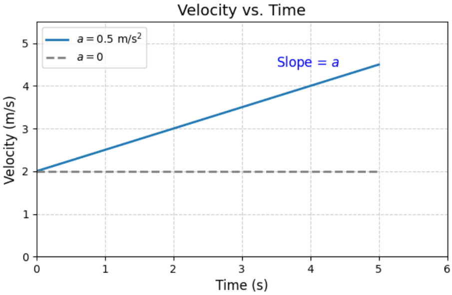
Example: You walk down the train platform at a steady pace. The slope of your position graph is constant, and the velocity graph's rectangular area gives how far you walked.

Definition of acceleration, and express of velocity at time t from acceleration:

$$a = \frac{\Delta v}{\Delta t} = \frac{v(t) - v(t_\circ)}{t - t_\circ} \longrightarrow v(t) = v(t_\circ) + a(t - t_\circ) \quad \Rightarrow \quad \boxed{v(t) = v_0 + at}$$

Velocity changes linearly with time. Equation simplifies when choosing $t_\circ = 0s$.

$$v(t) = v_0 + at$$



Consider this example: you are standing at the train station, your friend is walking through the train looking for an empty cabin, while the train is taking off (accelerating).

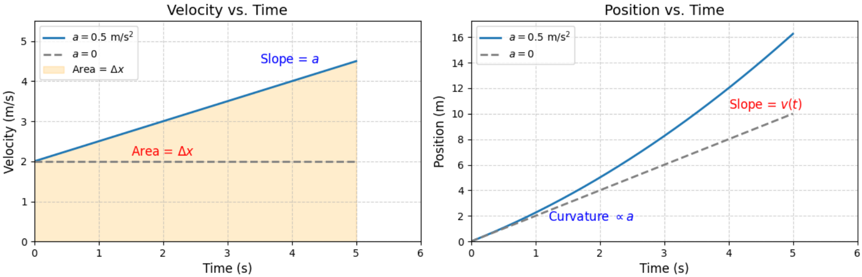
$v_0 \longrightarrow$ velocity of your friend (horizontal dashed line)

$at \longrightarrow$ velocity of the train

Connecting Position, Velocity, and Constant Acceleration

When $a = 0$, we have: $x(t) = x_0 + v_0 t$.

When $a \neq 0$, the position is a **quadratic** function of time $\boxed{x(t) = x_0 + v_0 t + \frac{1}{2} a t^2}$

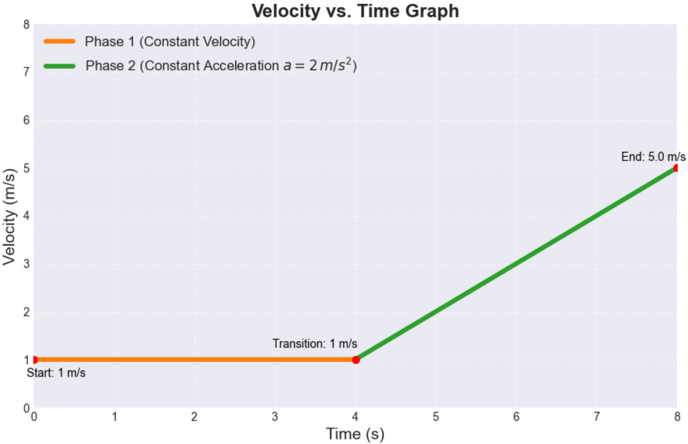


The meaning of each term:

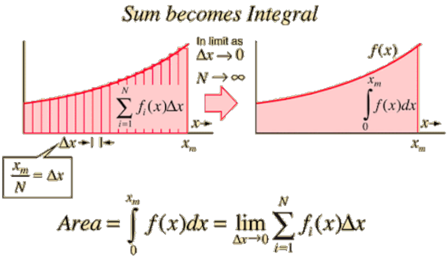
$x_0 \longrightarrow$ initial position (just a constant)
$v_0 t \longrightarrow$ distance travelled due to initial velocity (area of a rectangle on $v(t)$ graph)
$\frac{1}{2} a t^2 \longrightarrow$ distance travelled due to acceleration (area of a triangle on $v(t)$ graph)

The **area** under the $v(t)$ vs. t graph gives Δx.

If acceleration changes over time, brake the velocity graph into segments and add up the areas:
First part (orange): $\Delta x = v_1 \Delta t_1$ (time goes from 0 s to 4 s)
Second part(green) $\Delta x = v_1 \Delta t_2 + \frac{1}{2} a \Delta t_2^2$ (time goes from 4 s to 8 s)



If the acceleration is time-dependent, the segments of the velocity curve are not straight lines, hence you can't calculate the area under the $v(t)$ graph by simple triangles and rectangles. The area will have to be calculated with integrals:



Position is the **integral** of velocity (since it's the area under the velocity curve):

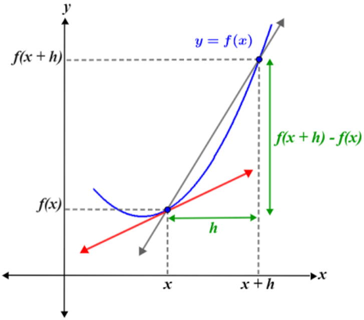
$$x(t) = \int v(t)\, dt$$

Velocity is the **integral** of acceleration (since it's the area under the acceleration curve):

$$v(t) = \int a(t)\, dt$$

Velocity and Acceleration as Derivatives

Similarly, the slope won't be simply $v = \Delta x/\Delta t$ and $a = \Delta v/\Delta t$ but instead we need differentials:



Velocity is the **derivative** of position:

$$\boxed{v(t) = \frac{dx}{dt}}$$

Acceleration is the **derivative** of velocity:

$$\boxed{a(t) = \frac{dv}{dt}}$$

Rules of Differentiation for Polynomials

For a general polynomial function:

$$f(t) = a_0 t^0 + a_1 t^1 + a_2 t^2 + a_3 t^3 + \cdots + a_n t^n = \sum_i a_i t^i$$

The **derivative** (rate of change) is obtained by multiplying by the exponent and reducing its power by one:

derivative of a single polynomial term: $\boxed{\frac{d}{dt}[t^n] = nt^{n-1}}$

the full polynomial: $\Rightarrow f'(t) = a_1 + 2a_2 t + 3a_3 t^2 + \cdots + na_n t^{n-1}$

Examples of Differentiating Polynomial Functions

Example 1: Second-Order Polynomial

$$f(t) = 4t^2 + 3t + 2$$
$$\frac{df}{dt} = 8t + 3$$

The derivative of t^2 is $2t$, and the derivative of t is 1.
Constant terms disappear when differentiating.
Geometrically: the slope of the parabola $f(t)$ changes linearly with t.
E.g. if f(t) is the displacement, then the velocity at $t = 2\text{s}$ is $v(t) = 19\text{m/s}$

Example 2: Third-Order Polynomial

$$g(t) = 2t^3 - 5t^2 + 3t - 4$$
$$\frac{dg}{dt} = 6t^2 - 10t + 3$$

The derivative of t^3 is $3t^2$, and of t^2 is $2t$.
The slope of a cubic function varies quadratically with t.
For physics: if $g(t)$ is position, then $\frac{dg}{dt}$ is velocity, changing nonlinearly in time.
E.g. if g(t) is the displacement, then the velocity at $t = 2\text{s}$ is $v(t) = 7\text{m/s}$

Rules of Integration for Polynomials

The **integral** (accumulated area) increases the power by one and divides by the new exponent (it's the opposite of differentiation):

$$\int t^n \, dt = \frac{t^{n+1}}{n+1} + C$$

$$\Rightarrow \int f(t) \, dt = a_0 t + \frac{a_1 t^2}{2} + \frac{a_2 t^3}{3} + \cdots + \frac{a_n t^{n+1}}{n+1} + C$$

Example:

$$\text{function: } f(t) = 3t^2 + 2t + 1$$

$$\text{derivative: } f'(t) = 6t + 2$$

$$\text{integral: } \int f(t) \, dt = t^3 + t^2 + t + C$$

The **definite integral** gives the accumulated change (area) between two times:

$$\boxed{\int_{t_1}^{t_2} t^n \, dt = \frac{t_2^{\,n+1} - t_1^{\,n+1}}{n+1}}$$

For a general polynomial:

$$f(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$$

$$\Rightarrow \int_{t_1}^{t_2} f(t) \, dt = a_0(t_2 - t_1) + \frac{a_1}{2}(t_2^2 - t_1^2) + \frac{a_2}{3}(t_2^3 - t_1^3) + \cdots + \frac{a_n}{n+1}(t_2^{\,n+1} - t_1^{\,n+1})$$

Example: Integrating Velocity to Find Displacement

Suppose the velocity of an object changes with time as (same as on previous slide):

$$v(t) = 3t^2 + 2t + 1 \quad [\mathrm{m/s}]$$

To find how far the object travels starting at $t = 0$, we integrate velocity over time:

$$x(t) = \int_0^t v(t)\,dt = t^3 + t^2 + t - (0 + 0 + 0)$$

Then, for example:

$$x(1) = 1^3 + 1^2 + 1 = 3 \text{ m}$$
$$x(2) = 2^3 + 2^2 + 2 = 14 \text{ m}$$
$$x(3) = 3^3 + 3^2 + 3 = 39 \text{ m}$$

The displacement between $t = 1$ s and $t = 3$ s is:

$$\Delta x = x(3) - x(1) = 39 - 3 = 36 \text{ m}$$

Interpretation: The area under the $v(t)$ curve between 1 s and 3 s equals 36 m — this is the total distance traveled during that interval.

Differentiation and Integration are Inverse Operations

Consider the polynomial:

$$f(t) = 3t^2 + 2t + 1$$

Its integral is:

$$\int f(t)\,dt = t^3 + t^2 + t + C$$

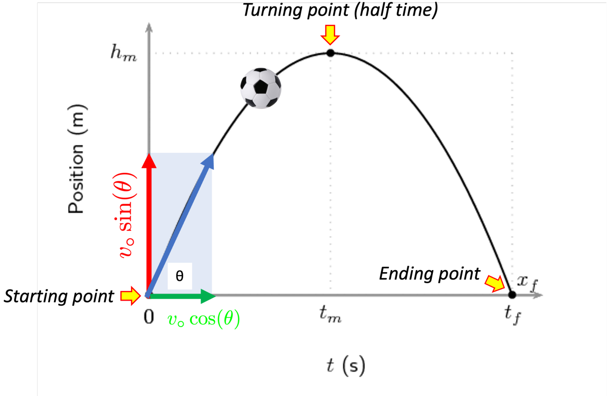
Differentiating the result gives back the original function:

$$\frac{d}{dt}\left(t^3 + t^2 + t + C\right) = 3t^2 + 2t + 1 = f(t)$$

Conclusion: Differentiation and integration are **inverse operations**. The derivative of an integral returns the original function, and the integral of a derivative returns the original function (up to a constant).

Projectile Motion: Ball Kicked at an Angle

A soccer ball is kicked with initial speed v_0 at an angle θ above the horizontal.



Questions we will answer:

What is the total time the ball remains in the air?
How far from the starting point will the ball land?
What is the maximum height the ball reaches before descending?
What is the final velocity when the ball hits the ground?

Projectile Motion: Ball Kicked at an Angle

Motion will take place in a plane mapped out by the initial velocity vector and the acceleration vector (due to gravity). I.e. the motion is in 2 dimensions.

Define the origin to be the starting point. Measure the angle θ relative to the x-axis.

Horizontal and vertical components of velocity at t=0:

$$v_{0x} = v_0 \cos\theta, \qquad v_{0y} = v_0 \sin\theta$$

Gravity is only affecting v_y. Thus, the displacement as a function of time:

$$x(t) = v_{x0} t,$$

$$\boxed{y(t) = v_{y0} t + \frac{1}{2} a t^2 = v_{y0} t - \frac{1}{2} g t^2}$$

In the above we used that the acceleration is $a = -g$. Numerically, $g = 9.81 m/s^2$.

Projectile Motion: Ball Kicked at an Angle

The **time of flight:** can be calculated by setting y(t)=0. I.e. we look for the time at which the y coordinate is zero, i.e. the ball is at the ground level.

$$y(t) = v_{y0}t - \frac{1}{2}gt^2 = 0$$

This is is quadratic in t, thus there will be two solutions (t_{start} and t_{flight}).
A quick way to solve it is to first rewrite the equation as:

$$y(t) = (v_{y0} - \frac{1}{2}gt)t = 0$$

A product is zero if either one of the factors are zero. This gives t=0 as a solution (this is trivial) and

$$(v_{y0} - \frac{1}{2}gt) = 0 \longrightarrow \boxed{t_{\text{flight}} = \frac{2v_{y0}}{g}}$$

Maximum height occurs at mid-flight. So evaluate $y(t)$ at $t = t_{\text{flight}}/2$:

$$\boxed{y_{\text{max}} = \frac{v_{y0}^2}{2g}}$$

Horizontal range:

$$x_{\text{final}} = v_{x0}\, t_{\text{flight}} \longrightarrow \boxed{x_{\text{final}} = \frac{2v_{x0}v_{y0}}{g}}$$

Projectile Motion: Ball Kicked at an Angle

Trajectory shape (eliminate time):

We have the x(t) and y(t) functions:

$$x(t) = v_{x0}t,$$

$$y(t) = v_{y0}t + \frac{1}{2}at^2 = v_{y0}t - \frac{1}{2}gt^2$$

To map out the trajectory we need $y(x)$! So, we need to eliminate t from $x(t)$ and $y(t)$ and express y as a function of x.

The simplest way to do it is to express t from $x(t)$ and then plug that into $y(t)$.

$$x(t) = v_{x0}t \longrightarrow t = x(t)/v_{x0}$$

Now plug this into:

$$y(t) = v_{y0}t - \frac{1}{2}gt^2$$

We find:

$$y(x) = \frac{v_{y0}}{v_{x0}}x - \frac{gx^2}{2v_{x0}^2}$$

The motion is a **parabola**.

Like: $f(x) = b + 0 + b_1x + b_2x^2$ with $b_0 = 0$, $b_1 = \frac{v_{y0}}{v_{x0}}$ and $b_2 = -\frac{g}{2v_{x0}^2}$.

Projectile Motion: Ball Kicked at an Angle

The final velocity?

v_x will always be the same, since there is no acceleration in the x-direction.

v_y can be found by plugging $t_{flight} = \frac{2v_{y0}}{g}$ into $v_y(t) = v_{y0} - gt$

$$v_y(t_{\text{flight}}) = v_{y0} - g\left(\frac{2v_{y0}}{g}\right)$$

Simplify:

$$v_y(t_{\text{flight}}) = v_{y0} - 2v_{y0} = -v_{y0}$$

At the end of the flight, the vertical velocity has the same magnitude but opposite direction as it was initially.

Since $v = \sqrt{v_x^2 + v_y^2}$, the final velocity is the same as the initial velocity.

Projectile Motion (L2): Ball Falling Below Initial Ground Level

Assume this time the person kicking the ball is standing on top of a cliff that's 10m high.

(a) How far will the ball go?

(b) What will be the time of flight?

(c) What will be the final velocity?

Projectile Motion (L2): Ball Falling Below Initial Ground Level

(a) How far will the ball go?

The solution hinges on the fact that the trajectory will be exactly the same!

$$y(x) = \frac{v_{y0}}{v_{x0}} x - \frac{g x^2}{2 v_{x0}^2}$$

The above relationship connects y_{end} and x_{end} too, since it's is on the trajectory:

$$y_{\text{end}} = \frac{v_{y0}}{v_{x0}} x_{\text{end}} - \frac{g x_{\text{end}}^2}{2 v_{x0}^2} \Rightarrow -\frac{g}{2 v_{x0}^2} x_{\text{end}}^2 + \frac{v_{y0}}{v_{x0}} x_{\text{end}} - y_{\text{end}} = 0$$

This is a quadratic equation for x. For a quadratic equation $ax^2 + bx + c = 0$, the solutions are

$$\boxed{x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

.
We only care about the positive solution, since the ball is flying in the positive x-direction, so:

$$\boxed{x_{\text{end}} = \frac{v_{x0}}{g} \left(v_{y0} + \sqrt{v_{y0}^2 - 2 g y_{\text{end}}} \right)}$$

Check: when $y_{\text{end}} = 0$ (like in the previous example), then $x_{\text{end}} = \frac{2 v_{x0} v_{y0}}{g}$

(b) What will be the time of flight?

We can just plug either y_end into $y(t)$ or x_end into $x(t)$ and solve.

Let's work in the y-direction:

$$y_{\text{end}} = v_{y0}\, t_{\text{fight}} - \frac{1}{2} g t_{\text{flight}}^2$$

This is a quadratic equation for t, and (again) we only care about the positive root:

$$\boxed{t_{\text{flight}} = \frac{v_{y0} + \sqrt{v_{y0}^2 - 2g y_{\text{end}}}}{g}}$$

Check: if $y_{\text{end}} = 0$, then $t_{\text{flight}} = \frac{2v_{y0}}{g}$, as it should.

(c) What will be the final velocity?

v_x will always be the same, since there is no acceleration in the x-direction.

v_y can be found by plugging t_{flight} into $v_y(t) = v_{y0} - gt$

$$\boxed{v_y = -\sqrt{v_{y0}^2 - 2gy_{\text{end}}}}$$

Check: if $y_{\text{end}} = 0$ we get $v_y = -v_{y0}$, as we should.

The total velocity is then $v = \sqrt{v_{x0}^2 + v_{y0}^2 - 2gy_{\text{end}}}$

Notice:

- If $y_{\text{end}} < 0$, the ball lands below its starting point and thus strikes the ground with a higher speed.
- If $y_{\text{end}} > 0$, the ball lands above its starting point (for example, when kicked up onto a cliff) and therefore hits with a lower speed.